International Journal of Modern Physics A
Vol. 36, No. 30 (2021) 2150215 (11 pages)

DOI: 10.1142/S0217751X21502158

Relativistic free fermions in spiral dislocation space–time with a distortion of a radial line into a spiral

S. Zare* and H. Hassanabadi†

Faculty of Physics, Shahrood University of Technology, Shahrood, Iran
**soroushzrg@gmail.com*
†h.hasanabadi@shahroodut.ac.ir

G. Junker

European Southern Observatory, Karl-Schwarzschild-Straße 2,
85748 Garching, Germany
gjunker@eso.org

Received 3 July 2021
Revised 28 September 2021
Accepted 29 September 2021
Published 12 November 2021

Relativistic quantum mechanics of free fermions in the presence of the spiral dislocation of space–time with a distortion of a radial line into a spiral is studied within the Katanaev–Volovich geometric approach. The generalized Dirac equation in this background is constructed. Exact closed-form solutions are found by reducing the problem to that of a nonrelativistic two-dimensional $1/r$-problem with a complex coupling constant. The influence of the defect parameter related to the spiral dislocation on these solutions is investigated. We also study the charge density of free fermions in the presence of such a spiral dislocation in space–time. Based on the Bender–Boettcher approach for non-Hermitian Hamiltonians we study, in addition, bound-state solutions of the system.

Keywords: Topological defects; Spiral dislocations; Dirac equation; Bender–Boettcher approach; PT symmetry.

PACS numbers: 03.65.Ge, 03.65.−w, 03.65.pm, 02.40.−k

1. Introduction

One of the predictions of grand unified theories is that topological defects might be generated through phase transitions caused by vacuum symmetry breaking in the early universe. In recent decades, the influences of such topological defects on the physical properties of various systems have been investigated (for some reviews see Refs. 1–3). The existence of linear topological defects in gravitation

†Corresponding author.

and condensed matter physics can be related to curvatures (disclinations), torsions (dislocations), and a combination of them called dispirations.[4–6] Thus, disclinations and dislocations (classified as screw and spiral dislocations) in the elastic medium as well as cosmic defects can be viewed as some sort of background deformations (linear defects) in space–time. Hence, many researchers have studied the influence of these topological defects on the nonrelativistic and relativistic quantum behavior from various different aspects.[7–9] Among them let us mention the investigation of the Aharonov–Bohm effect,[10,11] interactions of the electric and magnetic quadrupole moment with topological defects,[12,13] the quantum harmonic oscillator,[14] the Dirac oscillator,[15] relativistic fermions and bosons,[11,16] the study of the Kaluza–Klein theory,[17] Landau levels,[18] spin and pseudospin symmetries[19] and the quantum dynamics in a cosmic string background.[20]

It is well known that nonrelativistic and relativistic particles can be studied in the framework of quantum mechanics, with the presence or absence of physical potentials, within the background of flat space–times and those with topological defects. The investigation of relativistic particles, depending on their internal spin degree of freedom, must be carried out within their mathematical framework, which are the Klein–Gordon equation, the Dirac equation and the Duffin–Kemmer–Petiau (DKP) equation for spin zero, spin one-half and spin one, respectively.[21–27] For the many works done in those frameworks we may mention the investigations of Dirac's equation in the presence of magnetic fields and noncentral electromagnetic potentials,[28,29] the interaction of Dirac fermions and the Dirac oscillator with topological defects,[15,16,19,24–26] the relativistic harmonic oscillator under pseudospin symmetry,[30] the scalar quantum particle with Gurses space–time,[31] the thermal properties of the one-dimensional boson particles in Rindler space–time,[32] the two-dimensional DKP oscillator in the presence of a Coulomb potential in the cosmic string background[33] and the DKP oscillator in the presence of topological defects.[27]

In this work, following the Katanaev–Volovich geometric approach,[9] we construct the generalized Dirac equation for relativistic spin one-half fermions in the presence of the spiral dislocation space–time with a distortion of a radial line into a spiral. Besides its spectral properties we also investigate the associated charge density behavior.

This paper is organized as follows. In Sec. 2, starting with the line element of the spiral dislocation in $(1 + 3)$-dimensional space–time, we present the associated metric tensor. Then, by using the geometric approach of Katanaev–Volovich, we construct the Dirac equation in the background of the spiral dislocation in space–time. Based on the underlying symmetries we make an appropriated ansatz and reduce the generalized Dirac equation to a Schrödinger-like equation for the radial Coulomb problem with a complex coupling constant. Then, in Sec. 3, we investigate the spectral properties of this non-Hermitian Schrödinger Hamiltonian which in turn allows us to discuss the spectral properties of the relativistic system under investigation. We also briefly discuss the charge density related to free fermions in the presence of the spiral dislocation in space–time. Section 4 reconsiders the

Schrödinger problem within the framework of the Bender–Boettcher (BB) approach of non-Hermitian PT-symmetric quantum mechanics. Finally, in Sec. 5, we give a short summary with some concluding remarks.

2. Dirac's Equation in Spirally Dislocated Space–Time

Motivated by the extensive work dedicated to the study of the interaction of non-relativistic and relativistic quantum particles within topological defect space–time as outlined above, we investigate here the influence of spiral dislocation with a distortion of a radial line into a spiral for the free Dirac fermion using the Katanaev–Volovich geometric approach.[9] Thus, by adopting units where $\hbar = 1$ and $c = 1$, the line element of the spiral dislocation with the mentioned distortion in $(1+3)$-dimensional space–time is given by[5,34]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left(1+\beta^2 r^2\right)\mathrm{d}r^2 + 2\beta r^2\,\mathrm{d}r\,\mathrm{d}\varphi + r^2\,\mathrm{d}\varphi^2 + \mathrm{d}z^2\,. \tag{2.1}$$

In the above the time coordinate is an arbitrary real number $t \in (-\infty, \infty)$. The radial coordinate $r = \sqrt{x^2+y^2}$, the azimuthal angle φ and altitude z are the usual cylindric coordinates in the three-dimensional space with ranges given by $[0,\infty)$, $[0, 2\pi)$ and $(-\infty,\infty)$, respectively. In (2.1) the topological defect parameter is denoted by $\beta \geq 0$ and is related to Burger's vector $\mathbf{b}$ through the relation $\beta = |\mathbf{b}|/2\pi$. The direction of Burger's vector relative to the defect line determines the type of the defect in disordered solids.

The contravariant metric tensors $\mathrm{g}^{\mu\nu}$ associated with the line element (2.1) are given in the form

$$\mathrm{g}^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -\beta & 0 \\ 0 & -\beta & \beta^2 + \dfrac{1}{r^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mu,\nu \in \{t, r, \varphi, z\}\,. \tag{2.2}$$

To study the relativistic quantum mechanics associated with Dirac particles in the presence of topological defect space–time, we construct a local reference frame related to defected space–time, in a manner similar to the well-known procedure used in curved space–time, and redefined the Dirac spinors in this background.[26,35,36] The local reference frame may be defined via $\hat{\theta}^a = e^a{}_\mu(x)\mathrm{d}x^\mu$, where the objects $\mathrm{d}x^\mu$ and $e^a{}_\mu(x)$ are introduced as coordinate basis of one-forms and tetrads, respectively. Moreover, the tetrads satisfy the condition $g_{\mu\nu} = e^a{}_\mu(x)e^a{}_\nu(x)\eta_{ab}$, with the Minkowski tensor $\eta_{ab} = \mathrm{diag}(-\,+\,+\,+)$ and Latin indices $a, b \in \{0,1,2,3\}$. The inverse of tetrads is denoted by $e^\mu{}_a(x)$ satisfying the conditions $e^\mu{}_a(x)e^a{}_\nu(x) = \delta^\mu{}_\nu$ and $e^a{}_\mu(x)e^\mu{}_b(x) = \delta^a{}_b$.[16,19,37]

In the present case, let us introduce a local reference frame characterized by non-coordinate bases as follows $\hat{\theta}^0 = \mathrm{d}t$, $\hat{\theta}^1 = \mathrm{d}r$, $\hat{\theta}^2 = \beta r\,\mathrm{d}r + r\,\mathrm{d}\varphi$ and $\hat{\theta}^3 = \mathrm{d}z$. The nonvanishing inverse of tetrads corresponding to the line element (2.1) is written as ${e^t}_0 = {e^r}_1 = {e^z}_3 = 1$, ${e^\varphi}_1 = -\beta$ and ${e^\varphi}_2 = 1/r$.

To investigate spin one-half fermions under the background of a topological defect space–time using the Dirac equation, we must generalize this equation by using the covariant derivative ∇_μ replacing the partial derivative ∂_μ and taking into account the generalized Dirac matrices γ^μ instead of the standard Dirac matrices γ^a. Hence, the covariant derivative, under the background of the spiral dislocation space–time with a distortion of a radial line into a spiral, can be proposed as[38,39]

$$\nabla_\mu = \partial_\mu + \frac{\mathrm{i}}{4}\omega_{\mu ab}(x)\Sigma^{ab}\,, \quad \Sigma^{ab} = \frac{\mathrm{i}}{4}\left[\gamma^a, \gamma^b\right], \tag{2.3}$$

where the standard Dirac matrices corresponding to Minkowski space–time, denoted with γ^a, are given by[28,40–42]

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 1 \end{pmatrix}. \tag{2.4}$$

Here the Pauli matrices σ^i obey the relation $\left(\sigma^i\sigma^j + \sigma^j\sigma^i\right) = 2\eta^{ij}$ for indices i, $j = 1, 2, 3$. It should be noted that the generalized Dirac matrices, defined via relation $\gamma^\mu = e^\mu_a\gamma^a$, are found to be $\gamma^t = \gamma^0$, $\gamma^r = \gamma^1$, $\gamma^\varphi = -\beta\gamma^1 + \gamma^2/r$ and $\gamma^z = \gamma^3$. In addition, the second term to the right of Eq. (2.3) is defined as the spinorial connection denoted by Γ_μ and reads $\Gamma_\mu = \frac{\mathrm{i}}{4}\omega_{\mu ab}(x)\Sigma^{ab}$. The spin connection is denoted by $\omega_{\mu ab}(x)$ and its nonzero components, without considering the effect of torsion in this issue, are obtained via the Maurer–Cartan structure equations $\mathrm{d}\hat{\theta}^a + {\omega^a}_b \wedge \mathrm{d}x^\mu = 0$. Here, with the effect of torsion, we have ${\omega^a}_b = \omega_{\mu\ b}^{\ a}(x)\mathrm{d}x^\mu$ and hence $\omega_{\varphi\ 1}^{2} = -\omega_{\varphi\ 2}^{1} = 1$. Thus, according to resulting non-null components of spin connection, non-null components of the spinorial connection can be given by $\Gamma_\varphi = -\frac{\mathrm{i}}{2}\sigma_3$.

At this point, let us write the generalized form of the Dirac equation in this framework as[15,16,19,26,29]

$$\left[\mathrm{i}\gamma^\mu\nabla_\mu - M\right]\Psi(t, \mathbf{r}) = 0\,, \tag{2.5}$$

where M stands for the mass of the fermion and $\Psi(t, \mathbf{r})$ represents the corresponding Dirac field. To obtain a set of four coupled differential equations related to the generalized Dirac equation (2.5), we first substitute the covariant derivative (2.3) and the resulting Dirac matrices γ^μ together with the spinorial connection Γ_γ in Eq. (2.5). Then, based on the invariance of the system under time translations, translations along the z-axis as well as its invariance under rotation about the z-axis we propose an ansatz as follows:

$$\Psi(t, \mathbf{r}) = e^{-\mathrm{i}\mathcal{E}t + \mathrm{i}\left(\ell + \frac{1}{2}\right)\varphi + \mathrm{i}kz} \begin{pmatrix} \psi_1(r) \\ \psi_2(r) \\ \psi_3(r) \\ \psi_4(r) \end{pmatrix}, \tag{2.6}$$

where $\mathcal{E}$ denotes the energy eigenvalue of the fermionic system, $k \in (-\infty, \infty)$ is the wave number for the free motion along the z-axis and $j_z = \ell + 1/2$ is the z-component of total angular momentum with $\ell = 0, \pm 1, \pm 2, \ldots$. As a result we obtain a set of four differential equations related to the generalized Dirac equation (2.5) being expressed in the following form:

$$[\mathcal{E} - M]\psi_1 - k\psi_3 + \left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}r} + \left(\frac{\mathrm{i}}{r} + \beta\right)\left(\ell + \frac{1}{2}\right) + \frac{1}{2}\left(\frac{\mathrm{i}}{r} + \beta\right)\right]\psi_4 = 0\,, \quad (2.7\mathrm{a})$$

$$[\mathcal{E} - M]\psi_2 + \left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}r} - \left(\frac{\mathrm{i}}{r} - \beta\right)\left(\ell + \frac{1}{2}\right) + \frac{1}{2}\left(\frac{\mathrm{i}}{r} - \beta\right)\right]\psi_3 + k\psi_4 = 0\,, \quad (2.7\mathrm{b})$$

$$k\psi_1 + \left[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}r} - \left(\frac{\mathrm{i}}{r} + \beta\right)\left(\ell + \frac{1}{2}\right) - \frac{1}{2}\left(\frac{\mathrm{i}}{r} + \beta\right)\right]\psi_2 - [\mathcal{E} + M]\psi_3 = 0\,, \quad (2.7\mathrm{c})$$

$$\left[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}r} + \left(\frac{\mathrm{i}}{r} - \beta\right)\left(\ell + \frac{1}{2}\right) - \frac{1}{2}\left(\frac{\mathrm{i}}{r} - \beta\right)\right]\psi_1 - k\psi_2 - [\mathcal{E} + M]\psi_4 = 0\,. \quad (2.7\mathrm{d})$$

These explicit expressions suggest to establish a relation between the components of the Dirac spinor of the form

$$\psi_1(r) = \eta\psi_3(r)\,, \quad \psi_2(r) = \eta\psi_4(r)\,, \quad (2.8)$$

where the constant η is given by $\eta = \sqrt{(\mathcal{E} + M)/(\mathcal{E} - M)}$. Substitution of (2.8) in (2.7) results in a reduction of the four Dirac equations to a pair of Dirac equations, which, when combined, lead us to the following second-order radial equation for ψ_4

$$\frac{\mathrm{d}^2\psi_4(r)}{\mathrm{d}r^2} + \left[\frac{1}{r} - \mathrm{i}\beta(2\ell + 1)\right]\frac{\mathrm{d}\psi_4(r)}{\mathrm{d}r} + \left[\mathcal{E}^2 - M^2 - k^2 - \beta^2\ell(\ell + 1) - \frac{(\ell + 1)^2}{r^2}\right]\psi_4(r) = 0\,. \quad (2.9)$$

In the last step, we transform Eq. (2.9) into a solvable differential equation by setting

$$\psi_4(r) = e^{i\beta(2\ell+1)r/2}U(r)\,. \quad (2.10)$$

The resulting equation for U then reads

$$\frac{\mathrm{d}^2U(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}U(r)}{\mathrm{d}r} + \left[\mathcal{E}^2 - M^2 - k^2 + \frac{\beta^2}{4} + \frac{\mathrm{i}\beta}{2r}(2\ell + 1) - \frac{(\ell + 1)^2}{r^2}\right]U(r) = 0\,. \quad (2.11)$$

Obviously, this equation is identical in form with the two-dimensional radial Schrödinger equation for the Coulomb-like potential $V(r) = -\alpha/r$ with a purely imaginary coupling constant $\alpha = \mathrm{i}\beta(\ell + 1/2)$ with angular momentum $m = \ell + 1$ and energy eigenvalue $E = \mathcal{E}^2 - M^2 - k^2 + \beta^2/4$. Despite the fact that this does not represent a Hermitian Schrödinger operator we will show in the following section that it has real non-negative energy eigenvalues $E \geq 0$.

3. Spectral Properties and Charge Density

In this section we will first present explicit results for the eigenvalues and eigenfunction of the Schrödinger-like equation (2.11) by reducing it to the well-studied Whittaker equation. This will then provide us with closed-form expression of the spectral properties for the associated Dirac problem.

For this let us put $z := 2\mathrm{i}\kappa r$ and $w(z) = r^{1/2}U(r)$, which reduces Eq. (2.11) to the Whittaker equation

$$\left[\partial_z^2 - \frac{L^2 - 1/4}{z^2} + \frac{\nu}{z} - \frac{1}{4}\right] w(z) = 0\,, \tag{3.1}$$

where $L := |\ell + 1| \geq 0$, $\nu := \beta(2\ell + 1)/4\kappa$ and $\kappa^2 := E = \mathcal{E}^2 - M^2 - k^2 + \beta^2/4$. The Whittaker equation has the two independent solutions given by the Whittaker functions $M_{\nu,L}(z)$ and $W_{\nu,L}(z)$, see e.g. Ref. 43. Note that for $L > -1/2$ only the first one vanishes at $z = 0$ as $M_{\nu,L}(z) \sim z^{L+1/2}$ and is related to Kummer's confluent hypergeometric function ${}_1F_1(a, c, z)$ via relation

$$M_{\nu,L}(z) = z^{L+1/2} \mathrm{e}^{-z/2}\, {}_1F_1(L - \nu + 1/2, 2L + 1, z)\,. \tag{3.2}$$

Hence, the regular unbound solutions of (2.11) are given by

$$U_\kappa(r) = \mathcal{N} r^{|\ell+1|} \mathrm{e}^{-\mathrm{i}\kappa r}\, {}_1F_1(L - \nu + 1/2, 2L + 1, 2\mathrm{i}\kappa r)\,, \tag{3.3}$$

where $\mathcal{N}$ stands for a proper normalization constant and $\kappa = \pm\sqrt{E} \in \mathbb{R}$ denotes a radial incoming and outgoing wave, respectively. For a recent discussion on such Coulomb wave function see, for example, Ref. 44 and references therein. We only note that the above solution (3.3) is identical in form with the standard Coulomb wave function, however here $\nu \in \mathbb{R}$, whereas for the usual unbound Coulomb problem the parameter ν would have purely imaginary values.

Obviously the spectrum of the associated Dirac Hamiltonian is given by

$$\mathcal{E}_{k\kappa} = \pm\sqrt{M^2 + k^2 + \kappa^2 - \beta^2/4}\,, \quad k, \kappa \in \mathbb{R} \tag{3.4}$$

and is independent of the deformation parameter β. The components of the associated Dirac eigenstates read

$$\psi_{1_{k\kappa}}(r) = -\sqrt{\frac{\mathcal{E}_{k\kappa} + M}{\mathcal{E}_{k\kappa} - M}} \left[\frac{\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}r} + \left(\frac{\mathrm{i}}{r} + \beta\right)(\ell + 1)}{\sqrt{k^2 + \kappa^2} - k}\right] \mathrm{e}^{\mathrm{i}\beta(2\ell+1)r/2} U_\kappa(r)\,, \tag{3.5a}$$

$$\psi_{2_{k\kappa}}(r) = \sqrt{\frac{\mathcal{E}_{k\kappa} + M}{\mathcal{E}_{k\kappa} - M}}\, \mathrm{e}^{\mathrm{i}\beta(2\ell+1)r/2} U_\kappa(r)\,, \tag{3.5b}$$

$$\psi_{3_{k\kappa}}(r) = -\left[\frac{\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}r} + \left(\frac{\mathrm{i}}{r} + \beta\right)(\ell + 1)}{\sqrt{k^2 + \kappa^2} - k}\right] \mathrm{e}^{\mathrm{i}\beta(2\ell+1)r/2} U_\kappa(r)\,, \tag{3.5c}$$

$$\psi_{4_{k\kappa}}(r) = \mathrm{e}^{\mathrm{i}\beta(2\ell+1)r/2} U_\kappa(r)\,. \tag{3.5d}$$

In the flat space limit $\beta \to 0$ these reduce to the known solutions in cylinder coordinates[45] as in this case the function (3.3) in essence reduces to a Bessel function $U_\kappa(r) = \tilde{\mathcal{N}} J_{|\ell+1|}(\kappa r)$.

In the following we intend to investigate the charge density related to free fermions in the presence of the spiral dislocation in space–time. It is clear that in the context of the geometric approach, the four-current density, as a vector current, is expressed as a four-dimensional analog of the electric current density. Now, let us represent the zero-component of four-current components (charge density) corresponding to relativistic fermions as $J^t \propto \bar{\Psi}\gamma^t\Psi$, in which $\bar{\Psi} = \Psi^\dagger\gamma^0$, with $\Psi^\dagger = \left(\Psi^*\right)^T$, which means the complex conjugate transpose of the primary wave function Ψ. Thus, according to the results of Sec. 2, that is with regard to $\gamma^t = \gamma^0$ and Eqs. (2.6) and (2.7), we can write

$$J^0(r) = \bar{\Psi}\gamma^0\Psi = \Psi^\dagger\Psi\,. \tag{3.6}$$

Thereby

$$\begin{aligned} J^0_{k\kappa}(r) = &\left[1 + \frac{\left(\frac{(1+\ell)^2}{r^2} + \frac{\beta^2}{4}\right)}{\left(\sqrt{k^2+\kappa^2} - k\right)^2}\right]\left(1 + \frac{\mathcal{E}_{k\kappa} + M}{\mathcal{E}_{k\kappa} - M}\right)|U_\kappa(r)|^2 \\ &+ \frac{\left(1 + \frac{\mathcal{E}_{k\kappa}+M}{\mathcal{E}_{k\kappa}-M}\right)}{\left(\sqrt{k^2+\kappa^2} - k\right)^2}\left[U'_\kappa(r)U^{*\prime}_\kappa(r) + \left(\frac{2 + 2\ell + \mathrm{i}\beta r}{2r}\right)U'_\kappa(r)U^*_\kappa(r)\right. \\ &\left. + \left(\frac{2 + 2\ell - \mathrm{i}\beta r}{2r}\right)U_\kappa(r)U^{*\prime}_\kappa(r)\right]. \end{aligned} \tag{3.7}$$

If we focus on Eq. (3.7), we see that the charge density related to free fermions subject to spiral dislocation space–time with a distortion of a radial line into a spiral depends on wave functions $U_\kappa(r)$ given by Eq. (3.3), the β parameter corresponding to the spiral dislocation and the quantum numbers κ, k and ℓ.

4. The Bender−Boettcher Bound States

In this section, we reconsider the eigenvalue problem (2.11) following closely the discussion of BB.[46,47] With $x = r$ and $U(r) = x^{-1/2}F(x)$ this eigenvalue problem takes the form

$$F''(x) + [E - V_{\text{eff}}(x)]F(x) = 0\,, \tag{4.1}$$

where the effective one-dimensional potential is given by

$$V_{\text{eff}}(x) := \frac{(\ell+1)^2 - 1/4}{x^2} - \mathrm{i}\frac{\beta}{2x}(2\ell+1)\,. \tag{4.2}$$

For large x we may ignore the centrifugal part in this potential and then realize that $V(x) \sim (\mathrm{i}x)^{-1}$. Hence it behaves like the special case $N = -1$ of the BB potential $V_{\text{BB}}(x) \sim (\mathrm{i}x)^N$.[46,47] Following their approach we first extend the configuration space to the full real line $x \in \mathbb{R}$ keeping in mind that in the end we only consider solutions vanishing at $x = 0$. Obviously on this extended configuration space the above potential is PT-symmetry as $V^*_{\text{eff}}(x) = V_{\text{eff}}(-x)$ and hence the energy eigenvalues E are expected to be real. Furthermore, in order to have a well-defined eigenvalue problem with square-integrable eigenfunction we need to distort the configuration space into a contour in the lower complex x-plane approaching the anti-Stoke lines,[46,47] i.e.

$$\lim_{\mathrm{Re}\, x\to\pm\infty} \frac{x}{|x|} = \exp\left\{-\mathrm{i}\frac{\pi}{2} \pm \mathrm{i}\frac{2\pi}{N+2}\right\}. \tag{4.3}$$

In the current case where $N = -1$ this contour starts at $x = -\varepsilon - \mathrm{i}\infty$, approaches the origin and ends at $x = \varepsilon - \mathrm{i}\infty$ with $\varepsilon \searrow 0$.

As before let us put $z := 2\mathrm{i}\kappa r$ which, however, will now be real and non-negative along the distorted line $r \in \mathrm{i}\mathbb{R}^-$. Remember that we also have $\kappa^2 = E$ and $w(z) := F(x)$, which reduces Eq. (4.1) to the same Whittaker equation (3.1). Again the solution being regular at $z = 0$ is given by the Whittaker functions $M_{\nu,L}(z)$. However it diverges for large $z \to \infty$ unless $\nu - L - 1/2 \in \mathbb{N}_0$. Hence a square-integrable solution implies $\nu = n + L + 1/2$ with $n = 0, 1, 2, 3, \ldots$. This directly leads to the eigenvalues

$$E_{n\ell} = \frac{\beta^2}{4}\left(\frac{2\ell+1}{2n+2|\ell+1|+1}\right)^2, \tag{4.4}$$

which in turn result in the discrete spectrum of the associated Dirac problem given by

$$\mathcal{E}_{n\ell} = \pm\sqrt{\frac{(2\ell+1)^2\beta^2}{4(2n+1+2|\ell+1|)^2} - \frac{\beta^2}{4} + k^2 + M^2}\,. \tag{4.5}$$

Noting that $\kappa_{n\ell} = \beta(\ell+1/2)/(2n+2|\ell+1|+1)$, the corresponding eigenfunctions are given by

$$U_{n\ell}(r) = \mathcal{N}|r|^{|\ell+1|}\mathrm{e}^{-\kappa_{n\ell}|r|}\mathcal{L}_n^{2|\ell+1|}(2\kappa_{n\ell}|r|)\,, \tag{4.6}$$

with $\mathcal{N}$ being a normalization constant and $\mathcal{L}_n^{2L}(z)$ denotes the associated Laguerre polynomial of order $n = 0, 1, 2, 3, \ldots$. Recall that here the configuration space is given by $r \in \mathrm{i}\mathbb{R}^-$. The four components of the Dirac spinor corresponding to the primary wave function (2.6) in terms of wave functions (4.6) are in essence given via (3.5) with $\mathcal{E}_{k\kappa}$ replaced by $\mathcal{E}_{n\ell}$ and κ by $\kappa_{n\ell}$.

The eigenvalues (4.5) remain real for all values of β if

$$\frac{\beta^2}{M^2} \leq 4\left(1+\frac{k^2}{M^2}\right)\frac{(2n+1+2|\ell+1|)^2}{(2n+1+2|\ell+1|)^2-(2\ell+1)^2}\,. \tag{4.7}$$

Obviously for increasing n with a fixed ℓ the upper bound for β allowing for such a discrete eigenvalue decreases. This is illustrated in Fig. 1 where the influence of the spiral dislocation parameter β on the relativistic energy eigenvalues is shown.

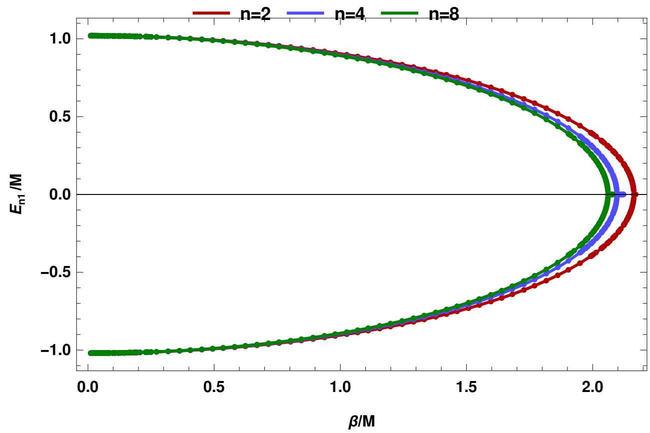

Fig. 1. The discrete energy eigenvalues (4.5) are shown in units of M as a function of the deformation parameter β/M parameter for the quantum number $n = 2, 4, 8$ as indicated. The other parameters are chosen to $k/M = 0.22$ and $l = 1$.

We also note that the eigenvalues (4.5) are bounded from below and above as follows:

$$M^2 + k^2 - \beta^2/4 \le \mathcal{E}_{n\ell}^2 \le M^2 + k^2 \,. \tag{4.8}$$

This may be compared with the continuous spectrum (3.4), which is bounded from below as follows:

$$\mathcal{E}_{k\kappa}^2 \ge M^2 + k^2 - \beta^2/4 \,. \tag{4.9}$$

In other words, the discrete eigenvalues are embedded in the continuous spectrum.

Again the problem is similar to the radial Coulomb problem with a purely complex interactions and represents a PT-symmetric system of a Bender–Boettcher-type for $N = -1$. Note that BB only considered $N > 1$ and state that for $N < 1$ there are no real values. However, for $N = -1$ we show the existence of such real eigenvalues. In addition, there is an accumulation point $\left[M^2 + k^2 - \beta^2/4\right]^{1/2}$ for large n, which is at the lower bound in contrast to the case for the usual Coulomb problem. For $\beta = 0$ the discrete spectrum vanishes as expected.

5. Conclusions

In this contribution we started with the presentation of spiral dislocation space–time with a distortion of a radial line into a spiral through the line element. In the light of the Katanaev–Volovich model, the interaction of relativistic free fermions with the spiral dislocation in space–time is studied through the Dirac equation.

In this regard, according to the resulting wave equation, we see that the background space–time described by spiral dislocations can determine a shifted Coulomb-like potential, that is, $\mathcal{C}_1 + \mathcal{C}_2/r$ in such a geometric approach. It should be noted that coefficients of shifted Coulomb-like potential are real and complex constants, i.e. $\mathcal{C}_1 = \beta^2/4$ and $\mathcal{C}_2 = \mathrm{i}\beta(\ell + 1/2)$ such that they consist of the defect parameter β and angular momentum quantum number ℓ. Accordingly, the investigation of the behavior of relativistic fermions under this background provides the shifted Coulomb-like potential generated by spiral dislocations in space–time so that this induced potential can be considered as an inevitable consequence of the existence of this type of the topological defect in background space–time.

It is worth noting that the analytical solutions corresponding to a fermion in the background involving the shifted Coulomb-like potential induced by a distortion of a radial line into a spiral in space–time, are obtained in the relativistic regime. From the components of the associated Dirac eigenstates, the effect of spiral dislocations in space–time can be observed on the behavior of relativistic fermions through the β parameter. Thus, by assuming $\beta \to 0$, we can find the relativistic wave function corresponding to a fermion in Minkowski space–time. In this way, it can be seen that the degeneracy of the relativistic energy spectrum is modified by the β parameter. The corresponding energy eigenvalues and wave functions are found by reducing this problem to the nonrelativistic $1/r$-problem in two dimensions with a purely imaginary coupling constant. The charge density is also briefly discussed. In addition, the complex $1/r$-problem is studied within the BB approach of PT symmetric quantum mechanics resulting in a discrete spectrum, which is discussed in some detail.

Acknowledgments

The authors thank the referee for a thorough reading of our paper and constructive suggestions.

References

1. T. W. B. Kibble, *Phys. Rep.* **67**, 183 (1980).
2. A. Vilenkin, *Phys. Rep.* **121**, 263 (1985).
3. G. de A Marques, C. Furtado, V. B. Bezerra and F. Moraes, *J. Phys. A: Math. Gen.* **34**, 5945 (2001).
4. R. A. Puntigam and H. H. Soleng, *Class. Quantum Grav.* **14**, 1129 (1977).
5. K. C. Valanis and V. P. Panoskaltsis, *Acta Mech.* **175**, 77 (2005).
6. A. Inomata, G. Junker and J. Reynolds, *J. Phys. A* **45**, 075301 (2012).
7. A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, 1994).
8. H. Kleinert, *Gauge Fields in Condensed Matter*, Vol. 2 (World Scientific, Singapore, 1989).
9. M. O. Katanaev and I. V. Volovich, *Ann. Phys.* **216**, 1 (1992).
10. C. Furtado, C. A. de Lima Ribeiro and S. Azevedo, *Phys. Lett. A* **296**, 171 (2002).
11. A. L. Cavalcanti de Oliveira and E. R. Bezerra de Mello, *Class. Quantum Grav.* **23**, 5249 (2006).
12. H. Hassanabadi, M. de Montigny and M. Hosseinpour, *Ann. Phys. (N.Y.)* **412**, 168040 (2020).
13. H. Hassanabadi, S. Zare, J. Kříž and B. C. Lütfüoğlu, *Europhys. Lett.* **132**, 60005 (2020).

14. C. Furtado and F. Moraes, *J. Phys. A*: *Math. Gen.* **33**, 5513 (2000).
15. J. Carvalho, C. Furtado and F. Moraes, *Phys. Rev. A* **84**, 032109 (2011).
16. S. Zare, H. Hassanabadi and M. de Montigny, *Eur. Phys. J. Plus* **135**, 122 (2020).
17. C. Furtado, F. Moraes and V. B. Bezerra, *Phys. Rev. D* **59**, 107504 (1999).
18. C. Furtado and F. Moraes, *Europhys. Lett.* **45**, 279 (1999).
19. S. Zare, H. Hassanabadi, G. J. Rampho and A. N. Ikot, *Eur. Phys. J. Plus* **135**, 748 (2020).
20. A. Inomata and G. Junker, *Phys. Lett. A* **376**, 305 (2012).
21. P. A. M. Dirac, *The Principles of Quantum Mechanics*, International Series of Monographs on Physics (Clarendon Press, 1981).
22. M. Moshinsky and A. Szczepaniak, *J. Phys. A*: *Math. Gen.* **22**, L817 (1989).
23. S. Bruce and P. Minning, *Nuovo Cimento A* **106**, 711 (1993).
24. E. R. Figueiredo Medeiros and E. R. Bezerra de Mello, *Eur. Phys. J. C* **72**, 2051 (2012).
25. F. M. Andrade and E. O. Silva, *Eur. Phys. J. C* **74**, 3187 (2014).
26. K. Bakke and C. Furtado, *Phys. Rev. A* **87**, 012130 (2013).
27. S. Zare, H. Hassanabadi and M. de Montigny, *Int. J. Mod. Phys. A* **35**, 2050195 (2020).
28. A. D. Alhaidari, H. Bahlouli and A. Jellal, *Int. J. Geom. Methods Mod. Phys.* **12**, 1550062 (2015).
29. A. D. Alhaidari, *Ann. Phys.* (*N.Y.*) **320**, 453 (2005).
30. R. Lisboa, M. Malheiro, A. S. de Castro, P. Alberto and M. Fiolhais, *Phys. Rev. C* **69**, 024319 (2004).
31. F. Ahmed, *Ann. Phys.* (*N.Y.*) **401**, 193 (2019).
32. A. Boumali and T. I. Rouabhia, *Phys. Lett. A* **385**, 126985 (2021).
33. A. Boumali and N. Messai, *Can. J. Phys.* **95**, 999 (2017).
34. A. Guvendi, S. Zare and H. Hassanabadi, *Eur. Phys. J. A* **57**, 192 (2021).
35. N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, UK, 1982).
36. S. Weinberg, *Gravitation and Cosmology*: *Principles and Applications of the General Theory of Relativity* (IE-Wiley, New York, 1972).
37. M. Nakahara, *Geometry, Topology and Physics* (Institute of Physics Publishing, Bristol, 1998), ISBN: 978-0750306065.
38. I. L. Shapiro, *Phys. Rep.* **357**, 113 (2002).
39. S. M. Carroll, *Spacetime and Geometry*: *An Introduction to General Relativity* (Cambridge University Press, 2019).
40. W. Greiner, *Relativistic Quantum Mechanics*: *Wave Equations*, 3rd. edn. (Springer, Berlin, 2000).
41. R. P. Martinez-y-Romero, H. N. Nunez-Yepez and A. L. Salas-Brito, *Eur. J. Phys.* **16**, 135 (1995).
42. G.-F. Wei and S.-H. Dong, *Phys. Lett. A* **373**, 49 (2008).
43. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th edn. (Cambridge University Press, Cambridge, UK, 1996).
44. D. Gaspard, *J. Math. Phys.* **59**, 112104 (2018).
45. B. Mihaila, J. F. Dawson and F. Cooper, *Phys Rev. D* **74**, 036006 (2006).
46. C. M. Bender and S. Boettcher, *Phys. Rev. Lett.* **80**, 5243 (1998).
47. F. Cannata, G. Junker and J. Trost, *Phys. Lett. A* **246**, 219 (1998).